Number Theory Class 1 - Solutions

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1 Review problems

1. $\boxed{23}$ Since the only rectangle Jon can arrange the desks in has width 1, he must have a prime number of students in his class. The only prime between 20 and 28 is 23.

2. Small composites

   a. $\boxed{9}$

   b. $\boxed{25}$

   c. $\boxed{49}$

   d. $\boxed{p^2}$ The smallest prime factor of the number we want must be p. Since the number must also be composite, we must multiply by a prime greater than or equal to p. To minimize the product, we should multiply by the smallest such number - p. Hence, the number is p^2.

3. 1 is a natural number that isn't prime, but neither is it composite. The textbook's definition makes it a composite.

4. Find the GCD of...

   a. Euclidean algorithm:

$$990=1 \cdot 720 + 270$$

$$720=2 \cdot 270 + 180$$

$$270=1 \cdot 180 + 90$$

$$180 = 2 \cdot \boxed{90} + 0$$

b. Euclidean algorithm:

$$819 = 1 \cdot 504 + 315$$

$$504 = 1 \cdot 315 + 189$$

$$315 = 1 \cdot 189 + 126$$

$$189 = 1 \cdot 126 + 63$$

$$126 = 2 \cdot \boxed{63} + 0$$

c. Euclidean algorithm:

$$25001 = 12 \cdot 2001 + 989$$

$$2001 = 2 \cdot 989 + 23$$

$$989 = 43 \cdot \boxed{23} + 0$$

d. The GCD of any set of primes is $\boxed{1}$.

6. Find the LCM of...

a. $18 = 2 \cdot 3^2$ and $42 = 2 \cdot 3 \cdot 7$. The LCM is $2 \cdot 3^2 \cdot 7 = \boxed{126}$.

b. $\boxed{63}$

c. $135 = 3^3 5$ and $144 = 2^4 3^2$. The LCM is $2^4 3^3 5 = \boxed{2160}$.

d. $\boxed{p_1 p_2 \cdots p_n}$

7. All contain a 3^n term with $n \geq 2$ and a 5^n term with $n \geq 1$.

8. $1 + 2 + 3 + \ldots + 70 = \frac{70 \cdot (70+1)}{2} = 35 \cdot 71 = 3 \cdot 5 \cdot \boxed{71}$.

9. Suppose the smaller is a perfect square. Then we multiply it by a square (4), which yields another square, but then we multiply it by a non-square (2). Multiplying a square by a non-square never yields a square. The other case is that the larger is a square. To get the smaller, we must divide by a square (4), which yields another square, but then we must divide by a non-square (2), and dividing a square by a non-square never yields a square.

10. The only way for the smaller of two numbers to be 8 times the larger is for them both to be negative, and negatives aren't squares.

2 Challenge problems

1. Let $x = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$. Suppose that $y \mid x$. Then m must be of the form $p_1^{b_1} p_2^{b_2} \dots p_n^{b_n}$, where, for all $i \in \mathbb{N}$ such that $1 < i < n$, $b_n \leq a_n$. Hence, b_i can be 0, 1, 2, ..., or a_i, so there are $a_i + 1$ possible exponents that p_i can take. Since the choice of exponent on one term is completely independent of the choice of exponent on another term, there are $(a_1 + 1)(a_2 + 1)(a_3 + 1) \cdots (a_1 + 1)$ possible values of y, and therefore $\boxed{(a_1 + 1)(a_2 + 1) \cdots (a_1 + 1)}$ divisors of x.

2. 5 and 7 are odd, so $5^2 3$ and $7^1 7$ are both odd. Adding them yields an even number, so the sum is divisible by $\boxed{2}$ - the smallest prime.

3. Suppose that a, m, and n are natural numbers, and that $GCD(a, n) = 1$. Explain why $GCD(m, n) = GCD(m, am + n)$.

Let $g = GCD(m, n)$, $c = \frac{m}{g}$, and $d = \frac{n}{g}$. Then $GCD(m, am + n) = GCD(gc, agc + gd) = g \cdot GCD(c, ac + d)$. Since $g = GCD(m, n)$, GCD(c,d)=1, and since $GCD(a, n) = 1$, $GCD(ac, d) = 1$. Hence, $GCD(c, ac + d) = 1$, so $GCD(m, am + n) = g \cdot 1 = GCD(m, n)$.